CS522 - Sample Final
Computational Tools and Methods for Finance

May 2005

Name: ____

1 Instructions

This exam contains 11 pages printed on one side only, and 7 problems. Please check now that you have a complete exam. If not, ask for another copy.

This exam is "closed books, closed notes." However, you can have a single page of formulas, but with no regular text on it. You can use a calculator, but it must be non-programmable and must not have in-built financial functions.

Your answers should be as clear, complete, and concise as possible. Try to eliminate any ambiguity in your answers. Write legibly - we will ignore answers that can not be reasonably deciphered. If you change your mind about an answer, make sure you cross out the parts that you do not want us to consider. Do not provide more than one answer to any question. If you do, we will randomly choose one version of the answer to grade, and we will ignore all the others.

If you need to make assumptions beyond those provided in the problems, feel free to do so, but state that you rely on such assumptions, provide them explicitly, and clearly explain why you needed them.

You are not allowed to cooperate with anybody else on this exam. All work submitted must be exclusively yours. Do not provide any kind of assistance to anybody else.

You have 150 minutes to work on the exam. Please plan to finish on time.

2 Financial Assumptions

1. Interest rates are positive.

2. There are no transactions costs, nor taxes. In addition, one can trade, lend or borrow unlimited amounts of any instrument, including cash. Neither interest rates, nor prices will change as a result of trading, lending, or borrowing by any individual. Fractional trading, lending or borrowing is possible.

3. There are no arbitrage opportunities.

4. All stock prices follow a log-normal distribution. No stocks pay dividends.

Good Luck!

1. Reproduce a given payoff [15 points]

   Consider the "asset or nothing" payoff defined below:

$$V(T,S) = \begin{cases} S, & if\ S > K, \\ 0, & otherwise. \end{cases}$$

   a Draw the graph of this payoff as a function of S.

   b Recreate this payoff by using a suitably chosen portfolio of instruments that have been discussed in class, possibly including non-standard options.

2. Properties of Options [15 points]

   a Consider a stock whose price S follows a log-normal distribution such that $\frac{dS}{S} = \mu dt + \sigma dW$, and assume that the variance of the respective distribution is 0. Assume that we also have a money account that pays a constant interest rate of r.

      (i) Establish a relationship between μ and r; justify it!

      (ii) What is the time-0 value of a call with strike K?

   b All else being equal, how will a decrease in volatility affect the value of a European put? What about the value of a European call? Prove your claims! Note: you can use any fact known from the course material to support your proof.

c Draw a qualitative graph representing the value of portfolio consisting of an American put and one share of the underlying stock, as a function of the underlying stock's price S. Mark all important points clearly and show any asymptotes unambiguosly.

3. Solutions for the Black-Scholes Equation [15 points]

Consider the two value functions given below ($\alpha \in R$, $t \in [0,T]$):

$$\begin{aligned} V_1(t,S) &= \alpha S \\ V_2(t,S) &= \alpha e^{rt} \end{aligned}$$

In answering the questions below, be as specific as you can be, and make sure you discuss all distinct cases that might arise.

a Using direct substitution, show that both these value functions satisfy the Black-Scholes differential equation.

b What instrument does V_1 correspond to? What is the value of Δ_1?

c What instrument does V_2 correspond to? What is the value of Δ_2?

4. Forward Rates [15 points]

   Consider a strictly downward sloping, continuous, smooth forward rate curve $f(t)$, $t \in [0, T]$ and fix $t_* \in [0, T]$.

   a Consider a zero-coupon bond with leftover maturity t_*. What is the relationship between the yield y_0 of the 0-coupon bond and $f(t_*)$? Prove your claim!

   b Consider a coupon bond with leftover maturity t_*. What is the relationship between the yield y of the coupon bond and $f(t_*)$? Prove your claim!

   c What is the relationship between y and y_0 as defined above? Prove your claim!

5. Simulating Asian Options [15 points]

   Consider an Asian option with strike K whose value at expiration is given by the following relation:

$$A(T, S) = \max\left[\frac{1}{2}\left(S(t) + S(T)\right) - K, 0\right],$$

   where t is an arbitrary, but fixed, intermediate time $t \in [0, T]$. Assume that the underlying stock price follows a log-normal distribution, further, assume that all relevant parameters are known.

   We have stated many times in class that simulating the stock price evolution from time 0 to time t and then from time t to time T, will not yield the same results.

   Choose an arbitrary stock price at time T, say, S^*.

   a Compute the probability $P(S(T) \leqslant S^* \mid S(0))$ of starting at time 0 from $S(0)$ and ending up with a stock price $S(T)$ no greater than S^*.

   Hint: Translate the stock price inequality that appears in the probability definitions into an inequality involving random normal variables.

b Compute the probability $P(S(T) \leqslant S^* \mid S(t))$ of starting at time t from $S(t)$ and ending up with a stock price $S(T)$ no greater than S^*.

c Prove that in general these probabilities will be different.

For partial credit: If you failed to compute the quantities requested above, argue qualitatively that these two probabilities are different. Make your statements are precise as possible!

6. Computing an Instrument's Value[10 points]

   Consider a European instrument that pays an amount $V(T, S) = S^n$ $(n > 0)$ at expiration. Compute the time t value of this instrument, where $t \in [0, T]$. You can assume that the stock price follows a log-normal distribution with the usual parameters.

   Hint: We have solved a similar problem when we derived the Black-Scholes formulas by using a direct evaluation of the discounted expected value of the payoff under the equivalent martingale probabilities.

7. Sparse Matrices [15 points]

Consider the sparse tri-diagonal matrix represented below:

$$M = \begin{bmatrix} a_1 & b_1 & 0 & \dots & 0 & 0 \\ c_1 & a_2 & b_2 & \dots & 0 & 0 \\ 0 & c_2 & a_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-1} & b_Y \\ 0 & 0 & 0 & \dots & c_X & a_n \end{bmatrix}.$$

This matrix is identical in structure with the matrix of the system of equations that results by applying the Crank-Nicholson method to the heat equation. To save space, we will linearize this matrix by storing it into a one-dimensional array L. The order of the values in L is $\begin{bmatrix} a_1 & c_1 & b_1 & a_2 & c_2 & b_2 & a_3 & c_3 & \dots & a_n \end{bmatrix}$. Both M and L should be treated as regular Matlab data structures; in particular you should assume that the smallest legal value of their respective indices is 1.

a What is the value of index X in the expression c_X that appears in the matrix? Also, what is the value of index Y in the expression b_Y that appears in the matrix?

b Let *linearize* be a Matlab function that takes matrix M as its input and returns array L. Implement function *linearize* below:

```
function L = linearize(M)
```

c Let *unpack* be a Matlab function that takes matrix L as its input and returns array M. Implement function *unpack* below:

```
function M = unpack(L)
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d Let *lookup* be a Matlab function that takes matrix L, and indices i and j, where i is an integer representing a row index, and j is an integer representing a column index. If the pair (i, j) represent a legal pair of indices into M, then function *lookup* must return $M(i, j)$. Otherwise (i.e. if (i, j) fall "outside" of M), the function must return NaN. Note: NaN is a shorthand for "not a number," and it is a value defined in the floating point standards to represent results of mathematically undefined operations. Finish the definition of function *lookup* whose header is given below:

```
function value = lookup(L, i, j)
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End of Exam